Problem 2.1

Show that the function

$$\psi(z,t) = (z+vt)^2$$

is a nontrivial solution of the differential wave equation. In what direction does it travel?

Solution

The aim is to show that the given function satisfies the wave equation.

$$\frac{\partial^2 \psi}{\partial t^2} \stackrel{?}{=} v^2 \frac{\partial^2 \psi}{\partial z^2}$$

Start taking derivatives, using the chain rule where appropriate.

$$\begin{aligned}
\frac{\partial \psi}{\partial t} &= \frac{\partial}{\partial t}[(z+vt)^2] = 2(z+vt)\cdot v = 2v(z+vt) \\
\frac{\partial^2 \psi}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial t}\right) = \frac{\partial}{\partial t}[2v(z+vt)] = 2v(v) = 2v^2 \\
\frac{\partial \psi}{\partial z} &= \frac{\partial}{\partial z}[(z+vt)^2] = 2(z+vt)\cdot 1 = 2(z+vt) \\
\frac{\partial^2 \psi}{\partial z^2} &= \frac{\partial}{\partial z}\left(\frac{\partial \psi}{\partial z}\right) = \frac{\partial}{\partial z}[2(z+vt)] = 2(1) = 2
\end{aligned}$$

Therefore,

$$\frac{\partial^2 \psi}{\partial t^2} = 2v^2 = (2)v^2 = v^2 \frac{\partial^2 \psi}{\partial z^2},$$

which means the given function for $\psi(z,t)$ is a solution of the wave equation.

$\psi(z,t) = (z+vt)^2$ travels in the positive z-direction if v is negative, and it travels in the negative z-direction if v is positive.